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NOTE
EDGE COLORING OF HYPERGRAPHS AND A CONJECTURE OF ERDŐS, FABER, LOVÁSZ

W. I. CHANG and E. L. LAWLER

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Call a hypergraph *simple* if for any pair u, v of distinct vertices, there is at most one edge incident to both u and v, and there are no edges incident to exactly one vertex. A conjecture of Erdős, Faber and Lovász is equivalent to the statement that the edges of any simple hypergraph on n vertices can be colored with at most n colors. We present a simple proof that the edges of a simple hypergraph on n vertices can be colored with at most $\lceil 1.5n-2 \rceil$ colors.

A conjecture of Erdős, Faber, Lovász ([1]—[3]) states that given n sets $A_1, A_2, \ldots, A_n$, with $|A_i|=n$ for $1\leqq i\leqq n$, and $|A_i\cap A_j|\leqq 1$ for $1\leqq i<j\leqq n$, one can color the elements of $\bigcup A_i$ with n colors so that for each i no two elements of A_i have the same color. This conjecture is now fifteen years old, and has inspired such work as [4] and [5]. We will present a simple proof that one can color the elements of $\bigcup A_i$ given $\lceil 1.5n-2 \rceil$ colors.

We first note that elements of $\bigcup A_i$ which belong to exactly one set A_i can be easily colored once all the remaining elements have been assigned colors. Given an Erdős, Faber, Lovász set system we remove all elements which belong to only one set A_i; we call such a modified set system *simple*. The Erdős, Faber, Lovász conjecture is then equivalent to the statement that any simple set system has an n-coloring. We propose to work instead in the following setting. Given a hypergraph $H=(V,E)$, let us define the *size* of an edge $e\in E$ as the number of vertices to which e is incident. We call H *simple* if

(∗) for any pair u, v of distinct vertices in H there is at most one edge incident to both u and v, and H contains no edges of size one.

Conjecture. *For any simple hypergraph H on n vertices, the chromatic index of H is at most n.*

Given a simple set system A_i for $1\leqq i\leqq n$, we can construct a hypergraph H on n vertices $V=\{1, 2, \ldots, n\}$ whose edges correspond to the elements of $\bigcup A_i$. More precisely, for each element e of $\bigcup A_i$ we assign to H an edge which is incident to each vertex i where $e\in A_i$. Clearly H is simple. Conversely, given a simple hypergraph $H=(\{1, 2, \ldots, n\}, E)$ we can construct a simple set system $A_i=\{e\in E \mid e$ is incident to vertex $i\}$ for $1\leqq i\leqq n$. In either construction a coloring of the elements of $\bigcup A_i$ corresponds trivially to a coloring of the edges of H. The following theo-

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rem implies $\lceil 1.5n-2 \rceil$ colors suffice for the element-coloring problem of Erdős, Faber, Lovász.

Theorem. *For any simple hypergraph H on n vertices, the chromatic index of H is at most* $\lceil 1.5n-2 \rceil$.

Proof. Arrange the edges of H in nonincreasing order of size. We will color the edges in this order, using $\lceil 1.5n-2 \rceil$ colors. Assume we next color an edge e of size $k \geqq 3$. At this point only edges of size k or greater have been assigned colors, so condition $(*)$ implies that at most $\lfloor (n-k)/(k-1) \rfloor$ of these can meet e at each of the k vertices to which e is incident. There will be an unused color for e if $k(n-k)/(k-1) < \lceil 1.5n-2 \rceil$, which holds for $k \geqq 3$. We color e using any such unused color. Assume we next color an edge (u, v) of size two. If there is any color unused at both u and v, we assign (u, v) that color. Otherwise we assert that there must be a vertex w for which the following holds:

(u, w) and (v, w) are edges of size two that have already been colored, the color of (u, w) is unused at v, and the color of (v, w) is unused at u.

We first examine the consequences of this assertion. We make the observations that at least $n/2$ colors are unused at each of u, v; that they are disjoint sets of colors, and that at most $n-1$ of them can be present at w. It follows that there is a color unused at w and at one of u, v. Without loss of generality assume there is a color c unused at both u and w. Then we can recolor (u, w) using c, leaving the original color of (u, w) unused at both u and v. Edge (u, v) can then be given that color. It remains for us to show the existence of such a vertex w. Let

$A = \{c | c$ is the color of a size-two edge incident to u but c is not the color of any size-three-or-greater edge incident to $v\}$

$B = \{c | c$ is the color of a size-two edge incident to v but c is not the color of any size-three-or-greater edge incident to $u\}$.

Then any color present at u or v but missing from $A \cup B$ must be the color of a size-three-or-greater edge incident to u or v. Condition $(*)$ implies $(n-|A|-2)/2$ and $(n-|B|-2)/2$ are upper bounds on the numbers of size-three-or-greater edges incident to u and v respectively. By assumption there are no colors unused at both u and v, so

$$1.5n-2.5 \leqq \lceil 1.5n-2 \rceil \leqq |A \cup B| + (n-|A|-2)/2 + (n-|B|-2)/2$$

or equivalently

$$n-1 \leqq |A \setminus B| + |B \setminus A|.$$

We note that $|A \setminus B|$ counts the number of edges (u, w) whose color is unused at v; similarly $|B \setminus A|$ counts the number of edges (v, w) whose color is unused at u. The existence of a suitable w then follows from the last inequality and a simple application of the pigeonhole principle. ∎

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W. I. Chang

Computer Science Division
571 Evans Hall
University of California
Berkeley, CA 94720, U.S.A.

E. L. Lawler

Computer Science Division
571 Evans Hall
University of California
Berkeley, CA 94720, U.S.A.